

Characteristics of Finite Jaco Graphs, $J_n(1), n \in \mathbb{N}$

(Johan Kok, Paul Fisher, Bettina Wilkens, Mokhwetha Mabula, Vivian Mukungunugwa)¹

Abstract

We introduce the concept of a family of finite directed graphs (*order 1*) which are directed graphs derived from a infinite directed graph (*order 1*), called the *1-root digraph*. The *1-root digraph* has four fundamental properties which are; $V(J_\infty(1)) = \{v_i | i \in \mathbb{N}\}$ and, if v_j is the head of an edge (arc) then the tail is always a vertex $v_i, i < j$ and, if v_k , for smallest $k \in \mathbb{N}$ is a tail vertex then all vertices $v_\ell, k < \ell < j$ are tails of arcs to v_j and finally, the degree of vertex k is $d(v_k) = k$. The family of finite directed graphs are those limited to $n \in \mathbb{N}$ vertices by lobbing off all vertices (and edges arcing to vertices) $v_t, t > n$. Hence, trivially we have $d(v_i) \leq i$ for $i \in \mathbb{N}$. We present an interesting Fibonacci-Zeckendorf result and present the Fisher Algorithm to table particular values of interest. It is meant to be an *introductory paper* to encourage exploratory research.

Keywords: Jaco graph, Directed graph, Jaconian vertex, Jaconian set, Number of edges, Shortest path, Fisher Algorithm, Zeckendorf representation

AMS Classification Numbers: 05C07, 05C12, 05C20, 11B39

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1 Introduction

We introduce the concept of a family of finite Jaco Graphs (*order 1*) which are directed graphs derived from the infinite Jaco Graph (*order 1*), called the *1-root digraph*. The *1-root digraph* has four fundamental properties which are; $V(J_\infty(1)) = \{v_i | i \in \mathbb{N}\}$ and, if v_j is the head of an edge (arc) then the tail is always a vertex $v_i, i < j$ and, if v_k , for smallest $k \in \mathbb{N}$ is a tail vertex then all vertices $v_\ell, k < \ell < j$ are tails of arcs to v_j and finally, the degree of vertex k is $d(v_k) = k$.

Definition 1.1. *The infinite Jaco Graph $J_\infty(1)$ is defined by $V(J_\infty(1)) = \{v_i | i \in \mathbb{N}\}$, $E(J_\infty(1)) \subseteq \{(v_i, v_j) | i, j \in \mathbb{N}, i < j\}$ and $(v_i, v_j) \in E(J_\infty(1))$ if and only if $2i - d^-(v_i) \geq j$.*

Definition 1.2. *The family of finite Jaco Graphs are defined by $\{J_n(1) \subseteq J_\infty(1) | n \in \mathbb{N}\}$. A member of the family is referred to as the Jaco Graph, $J_n(1)$.*

Definition 1.3. *The set of vertices attaining degree $\Delta(J_n(1))$ is called the Jaconian vertices of the Jaco Graph $J_n(1)$, and denoted, $\mathbb{J}(J_n(1))$ or, $\mathbb{J}_n(1)$ for brevity.*

Definition 1.4. *The lowest numbered (indiced) Jaconian vertex is called the prime Jaconian vertex of a Jaco Graph.*

Definition 1.5. *If v_i is the prime Jaconian vertex of a Jaco Graph $J_n(1)$, the complete subgraph on vertices $v_{i+1}, v_{i+2}, \dots, v_n$ is called the Hope subgraph of a Jaco Graph and denoted, $\mathbb{H}(J_n(1))$ or, $\mathbb{H}_n(1)$ for brevity.*

Definition 1.6. *If, in applying definition 1.1 to vertex v_i (not necessarily exhaustively), or for logical method of proof we have the edge (v_i, v_k) linked in a Jaco Graph $J_n(1)$, then the degree vertex v_i attains at v_k is called the, "at degree of v_i at v_k ", and is denoted, $d^*(v_i)@v_k$.*

Definition 1.7. *In $J_\infty(1)$ we have $n = d^+(v_n) + d^-(v_n)$ whilst in $J_n(1)$ we have $d(v_i) = \lceil d^+(v_i) \rceil + d^-(v_i), i \leq n$.*

Property 1: From the definition of a Jaco Graph $J_n(1)$, it follows that for the prime Jaconian vertex v_i , we have $d(v_m) = m$ for all $m \in \{1, 2, 3, \dots, i\}$.

Property 2: From the definition of a Jaco Graph $J_n(1)$, it follows that $\Delta(J_k(1)) \leq \Delta(J_n(1))$ for all $k \leq n$.

Property 3: The $d^-(v_k)$ for any vertex v_k of a Jaco Graph $J_n(1)$, $n \geq k$ is equal to $d(v_k)$ in the underlying Jaco Graph $J_k(1)$.

Lemma 1.1. *If in a Jaco Graph $J_n(1)$, and for smallest i with $d(v_i) = i$, the edge (v_i, v_n) is defined, then v_i is the prime Jaconian vertex of $J_n(1)$.*

Proof. If by definition 1.1 and for smallest i with $d(v_i) = i$ the edge (v_i, v_n) is defined, we have in the underlying graph of $J_n(1)$ that $d(v_j) \leq d(v_i)$ for all $j > i$. We also have that $d(v_s) < d(v_i)$, $s < i$. So it follows that $d(v_i) = \Delta(J_n(1))$ hence by definition 1.4 the vertex v_i is the prime Jaconian vertex of $J_n(1)$. \square

Lemma 1.2. *For all Jaco Graphs $J_n(1)$, $n \geq 2$ and, $v_i, v_{i-1} \in V(J_n(1))$ we have that in the underlying graph $|d(v_i) - d(v_{i-1})| \leq 1$.*

Proof. Consider the Jaco Graph $J_n(1)$, $n \geq 2$. The result is trivially true for all vertices $v_1, v_2, v_3, \dots, v_k$ if v_k is the prime Jaconian vertex of $J_n(1)$. Now consider the Hope subgraph $\mathbb{H}(J_n(1))$. All vertices of $\mathbb{H}(J_n(1))$ have equal degree so the result holds for the Hope subgraph *per se*. Furthermore if a vertex v_j , $(k+1) \leq j \leq n$ is linked to a vertex v_t , $1 \leq t \leq k$ then all vertices v_l , $(k+1) \leq l < j$ are linked to v_t which implies $|d(v_j) - d(v_l)| = 0 < 1$ hence $|d(v_{j+1}) - d(v_j)| \leq 1$. \square

Corollary 1.3. *For a Jaco Graph $J_n(1)$ the maximum degree $\Delta(J_n(1))$ might repeat itself as n increases to $n+1$, (i.e. $\Delta J_n(1) = \Delta J_{n+1}(1)$) but on an increase of we always obtain $\Delta J_{n+1}(1) = \Delta(J_n(1)) + 1$.*

Proof. The result follows from Lemma 1.2. \square

2 The Fisher Algorithm for $\{J_i(1), i \in \{4, 5, 6, \dots, s \in \mathbb{N}\}\}$

The family of finite Jaco Graphs are those limited to $n \in \mathbb{N}$ vertices by lobbing off all vertices (and edges arcing to vertices) $v_t, t > n$. Hence, trivially we have $d(v_i) \leq i$ for $i \in \mathbb{N}$.

Column 1 is the map: $\phi(v_i) \rightarrow i, \forall i$.

Column 2 is the in-degree of vertex v_i .

Column 3 is the out-degree of vertex v_i in $J_\infty(1)$.

Column 4 is the set $\mathbb{J}(J_i(1))$.

Column 5 is $\Delta(J_i(1))$.

Column 6 is the distance $d_{J_i(1)}(v_1, v_i)$.

We generally refer to the entries in a row i as: $ent_{1i} = i$, $ent_{2i} = d^-(v_i)$, $ent_{3i} = d^+(v_i)$, $ent_{4i} = \mathbb{J}(J_i(1))$, $ent_{5i} = \Delta(J_i(1))$, $ent_{6i} = d_{J_i(1)}(v_1, v_i)$ as interchangeable.

Note that rows 1, 2 and 3 follow easily from definition 1.1.

Step 0: Set $j = 4$, then set $i = j$ and $s \geq 4$.

Step 1: Set $ent_{1i} = i$.

Step 2: Set $ent_{2i} = ent_{1(i-1)} - ent_{5(i-1)}$. (Note that $d^-(v_i) = v(\mathbb{H}_{i-1}(1)) = (i-1) - \Delta(J_{i-1}(1))$).

Step 3: Set $ent_{3i} = ent_{1i} - ent_{2i}$. (Note that $d^+(v_i) = i - d^-(v_i)$).

Step 4: Consider $ent_{4(i-1)}$. If $ent_{4(i-1)} = \{v_k\}$, set $t = k$, else set $t = k + 1$.

Step 5: Set the prime Jaconian vertex as v_t so $\mathbb{J}(J_i(1)) = \{v_t\}$ to begin with. Let $l = t + 1, t + 2, \dots, i - 1$ and recursively calculate $i - ent_{1l} + ent_{2l}$. If $i - ent_{1l} + ent_{2l} = t$, add v_l to the set of Jaconian vertices, else go to Step 6.

Step 6: Set $ent_{5i} = t$. (Note that if $\mathbb{J}_i(1) = \{v_t, v_{t+1}, \dots, v_\ell\}$, then, $\Delta(J_i(1)) = t$).

Step 7: Select smallest k such that, $k + ent_{3k} \geq i$ then set $ent_{6i} = ent_{6k} + 1$.

Step 8: Set $j = i + 1$, then set $i = j$. If $i \leq s$, go to Step 1, else go to Step 9.

Step 9: Exit.

Proposition 2.1. *Consider the Jaco Graph $J_i(1)$, $i \geq 4$. If the Jaconian vertex of $J_{i-1}(1)$ is unique say, v_k then $k + d^+(v_k) < i$ and $(k + 1) + d^+(v_{k+1}) > i$.*

Proof. Because the Jaconian vertex v_k is unique to $J_{i-1}(1)$ it implies that edge (v_k, v_{i-1}) exists (see Theorem 2.11), so $k + d^+(v_k) = i - 1 < i$. And since edge (v_{k+1}, v_i) does not exist in $J_{i-1}(1)$ we have $d(v_{k+1}) = k - 1$.

By extending to $J_i(1)$ the edge (v_{k+1}, v_i) is linked. So degree of v_{k+1} increases to $d(v_{k+1}) = k$ implying $d^+(v_{k+1})$ increased by 1. Thus, $(k + 1) + d^+(v_{k+1}) = (k + 1) + (d^+(v_k) + 1) = (i - 1) + 2 = i + 1 > i$. \square

Lemma 2.2. *(Conjectured): If for $n \in \mathbb{N}$ we have that $d^+(v_n) = \ell$ is non-repetitive (meaning $d^+(v_{n-1}) < d^+(v_n) < d^+(v_{n+1})$) then, $\mathbb{J}(J_n(1)) = \{v_\ell\}$.*

Theorem 2.3. *(Morrie's Theorem): If a Jaco Graph $J_n(1)$, $n \geq 2$ has a prime Jaconian vertex v_k then:*

(a) $d^-(v_k) = d^-(v_{k+1})$ and $d^-(v_{k+2}) = d^-(v_{k+1}) + 1$ if and only if $\mathbb{J}(J_n) = \{v_k\}$ and

$$\mathbb{J}(J_{n+1}(1)) = \{v_k, v_{k+1}, v_{k+2}\},$$

(b) $d^-(v_k) = d^-(v_{k+1}) = d^-(v_{k+2})$ if and only if $\mathbb{J}(J_n) = \{v_k\}$ and $\mathbb{J}(J_{n+1}(1)) = \{v_k, v_{k+1}\}$.

Proof. Let $d^-(v_k) = d^-(v_{k+1})$ and $d^-(v_{k+2}) = d^-(v_{k+1}) + 1$ for the Jaco Graph $J_n(1), n \geq 2$. and let $\mathbb{J}(J_n(1)) = \{v_k\}$. From definition 1.7 and Steps 1, 2 and 3 of the Fisher Algorithm it follow that we have associated entries:

$$\begin{aligned} ent_{1k} &= k, ent_{2k} = d^-(v_k), ent_{3k} = d^+(v_k) = k - d^-(v_k) \text{ and,} \\ ent_{1(k+1)} &= k+1, ent_{2(k+1)} = d^-(v_{k+1}) = d^-(v_k), ent_{3(k+1)} = d^+(v_{k+1}) = (k+1) - d^-(v_k) \\ \text{and,} \\ ent_{1(k+2)} &= k+2, ent_{2(k+2)} = d^-(v_{k+2}) = d^-(v_k)+1, ent_{3(k+1)} = d^+(v_{k+2}) = (k+2) - d^-(v_k) - 1 \\ \text{and,} \\ ent_{1(k+3)} &= k+3, ent_{2(k+3)} = d^-(v_{k+3}) = d^-(v_k)+1, ent_{3(k+3)} = d^+(v_{k+3}) = (k+3) - d^-(v_k) - 1. \end{aligned}$$

Let $n = 2k - d^-(v_k)$ and it easily follows from Step 5 that $\mathbb{J}(J_n(1)) = \{v_k\}$. Now let $n = 2k - d^-(v_k) + 1$ and initialise $\mathbb{J}(J_{n+1}(1)) = \{v_k\}$ and set $t = k$. Also let $l = k+1, k+2, \dots, 2k - d^-(v_k)$.

For $l = k+1$ we have that $(2k - d^-(v_k) + 1) - (k+1) + d^-(v_k) = k = t$, so $v_{k+1} \in \mathbb{J}(J_{n+1}(1))$. For $l = k+2$ we have that $(2k - d^-(v_k) + 1) - (k+2) + d^-(v_k) + 1 = k = t$, so $v_{k+2} \in \mathbb{J}(J_{n+1}(1))$. For $l = k+3$ we have that $(2k - d^-(v_k) + 1) - (k+3) + d^-(v_k) + 1 = k - 1 \neq t$, so $v_{k+3} \notin \mathbb{J}(J_{n+1}(1))$.

So it follows that if $d^-(v_k) = d^-(v_{k+1})$ and $d^-(v_{k+2}) = d^-(v_{k+1}) + 1$ then $\mathbb{J}(J_n(1)) = \{v_k\}$ and $\mathbb{J}(J_{n+1}(1)) = \{v_k, v_{k+1}, v_{k+2}\}$ with $n \in \{2k - d^-(v_k), 2k - d^-(v_k) + 1\}$.

Conversely, if $\mathbb{J}(J_n(1)) = \{v_k\}$ and $\mathbb{J}(J_{n+1}(1)) = \{v_k, v_{k+1}, v_{k+2}\}$ we have from the inverse of definition 1.7 and Steps 1, 2, 3, 4, 5 and 6 of the Fisher Algorithm the associated entries:

$$\begin{aligned} ent_{1n} &= n, ent_{2n} = (n - k), ent_{3n} = k, ent_{4n} = \{v_k\}, ent_{5n} = k \Rightarrow \\ ent_{1k} &= k, ent_{2k} = 2k - n, ent_{3k} = n - k, ent_{5k} = k \Rightarrow \\ ent_{1(k+1)} &= k+1, ent_{2(k+1)} = 2k - n, ent_{3(k+1)} = n - k + 1, ent_{5(k+1)} = k \Rightarrow \\ ent_{1(k+2)} &= k+2, ent_{2(k+2)} = 2k - n + 1, ent_{3(k+2)} = n - k + 1, ent_{5(k+2)} = k+1. \\ \therefore d^-(v_k) &= d^-(v_{k+1}) \text{ and } d^-(v_{k+2}) = d^-(v_{k+1}) + 1. \end{aligned}$$

Result (b) follows similarly to (a). □

Proposition 2.4. For all Jaco Graphs $J_n(1)$, we have $\text{Card } \mathbb{J}(J_n(1)) \leq 3$.

Proof. It is evident that for some $m \in \mathbb{N}$, $\text{Card } \mathbb{J}(J_m(1)) = 3$. Let $\mathbb{J}(J_m(1)) = \{v_k, v_{k+1}, v_{k+2}\}$. So in Step 4 of the Fisher Algorithm we initially set $i = m$ and $t = k$. We also have that $i - (k + 2) + d^-(v_{k+2}) = t$.

From Morrie's theorem it follows that $d^-(v_k) = d^-(v_{k+1})$ and $d^-(v_{k+2}) = d^-(v_{k+1}) + 1$. It follows that, $d^-(v_{k+3}) = d^-(v_{k+1}) + 1$. However, in Step 5 we have $i - (k + 3) + d^-(v_{k+3}) = i - (k + 3) + d^-(v_{k+1}) + 1 = (i - (k + 2) + d^-(v_{k+2})) - 1 < t$. So vertex v_{k+3} cannot be added to $\mathbb{J}(J_m(1))$. \square

Corollary 2.5. *From Proposition 2.4 it follows that if and only if the Jaconian vertex of $J_{i-1}(1)$, $i \geq 2$ is unique say, v_k then $\mathbb{J}(J_i(1)) = \text{either } \{v_k, v_{k+1}\} \text{ or } \{v_k, v_{k+1}, v_{k+2}\}$.*

Proof. By extending from $J_{i-1}(1)$ to $J_i(1)$ the edge (v_k, v_i) is not linked. Because $d(v_{k+1}) = d(v_k) - 1$ in $J_{i-1}(1)$ and increases by 1 in $J_i(1)$ it follows that $d(v_{k+1}) = d(v_k)$ in $J_i(1)$. Hence, at least $\mathbb{J}_i(1) = \{v_k, v_{k+1}\}$. If $d^-(v_{k+2}) = d^-(v_{k+1}) + 1$, then $\mathbb{J}_i(1) = \{v_k, v_{k+1}, v_{k+2}\}$. So it follows that $\mathbb{J}_i(1) = \text{either } \{v_k, v_{k+1}\} \text{ or } \{v_k, v_{k+1}, v_{k+2}\}$.

Conversely, assume that $\mathbb{J}_i(1) = \text{either } \{v_k, v_{k+1}\} \text{ or } \{v_k, v_{k+1}, v_{k+2}\}$.

Case 1: Let $\mathbb{J}(J_i(1)) = \{v_k, v_{k+1}\}$. So $i - (k + 1) + d^-(v_{k+1}) = k$ in $J_i(1)$. Hence in $J_{i-1}(1)$ we have $(i - 1) - (k + 1) + d^-(v_{k+1}) = (i - (k + 1)) + d^-(v_{k+1}) - 1 = k - 1$. So from Step 5 of the Fisher Algorithm it follows that $v_{k+1} \notin \mathbb{J}(J_{i-1}(1))$. However, $v_k \in \mathbb{J}(J_{i-1}(1)) = \{v_k\}$.

Case 2: Let $\mathbb{J}(J_i(1)) = \{v_k, v_{k+1}, v_{k+2}\}$. Same reasoning as in case 1, follows. \square

Corollary 2.6. *If $k + d^+(v_k) = i$ and $(k + 1) + d^+(v_{k+1}) > i + 1$ then v_k is the unique Jaconian vertex of $J_i(1)$.*

Proof. The result follows directly from Step 5 of the Fisher Algorithm. \square

Proposition 2.7. *If we have $d^-(v_{k-1}) = d^-(v_k) = d^-(v_{k+1})$ then v_k is the unique Jaconian vertex of $J_l(1)$, $l = 2k - d^-(v_k)$.*

Proof. For $d^+(v_k) = k - d^-(v_k)$ and $l = k + d^+(v_k) = k + k - d^-(v_k) = 2k - d^-(v_k)$ it follows that v_k is a Jaconian vertex of $J_l(1)$. Furthermore, v_k is the unique Jaconian vertex of $J_l(1)$ because:

Case 1: For v_{k-1} and because $d^-(v_{k-1}) = d^-(v_k)$, we have $l - (k - 1) + d^-(v_k) = 2k - d^-(v_k) - (k - 1) + d^-(v_k) = 2k - d^-(v_k) - k + 1 + d^-(v_k) = 2k - k + 1 = k + 1 > k$. Hence, v_{k-1} is not a Jaconian vertex of $J_l(1)$.

Case 2: For v_{k+1} and because $d^-(v_{k+1}) = d^-(v_k)$, we have $l - (k + 1) + d^-(v_k) = 2k - d^-(v_k) - (k + 1) + d^-(v_k) = 2k - d^-(v_k) - k - 1 + d^-(v_k) = 2k - k - 1 = k - 1 < k$. Hence, v_{k+1} is not a Jaconian vertex of $J_l(1)$. \square

Proposition 2.8. $\mathbb{J}(J_{k-1}(1)) = \{v_{l-1}\}$ if and only if $d^+(v_k) = d^+(v_{k+1}) = l$.

Proof. If $\mathbb{J}(J_{k-1}(1)) = \{v_{l-1}\}$, implying $\Delta(J_{k-1}(1)) = l - 1$, it follows from Step 2 of the Fisher Algorithm that $d^-(v_k) = (k - 1) - \Delta(J_{k-1}) = (k - 1) - d^+(v_{k-1})$. So because $k = (k - 1) + 1$, it follows that $d^+(v_k) = l$ because $d(v_i) = d^+(v_i) + d^-(v_i), \forall i$. By similar reasoning $d^+(v_{k+1}) = l$, so it holds that $d^+(v_k) = d^+(v_{k+1}) = l$.

Conversely, if $d^+(v_k) = d^+(v_{k+1}) = l$, it follows by inverting the convergence properties of the Fisher Algorithm that $\mathbb{J}(J_{k-1}(1)) = \{v_{l-1}\}$. \square

Theorem 2.9. Let $m = n + \Delta(J_n(1))$, then $\Delta(J_m(1)) = \text{either } n \text{ or } n - 1$.

Proof. Let $m = n + \Delta(J_n(1))$.

Case 1: Assume $\mathbb{J}(J_n(1)) = \{v_k\}$ so $\Delta(J_n(1)) = k$. We have that $d^+(v_n) = k$ so in $J_m(1)$ the edge (v_n, v_m) is defined to attain $d(v_n) = n$, and v_n is the prime Jaconian vertex of $J_m(1)$, $m = n + \Delta(J_n(1))$. Hence, $\Delta(J_m(1)) = n$.

Case 2: Assume $\mathbb{J}(J_n(1)) = \{v_k, v_{k+1}\}$ or $\{v_k, v_{k+1}, v_{k+2}\}$. We have that $d^+(v_n) = k$ or $k + 1$. So by the same reasoning as in Case 1 it follows that $\Delta(J_m(1)) = n$ or $n - 1$. \square

Theorem 2.10 (Conjectured). For the Jaco Graphs $J_n(1)$, $J_m(1)$ with $n \geq 3$, $m \geq 3$, $n \neq m$ we have

$$\Delta(J_{n+m}(1)) = \begin{cases} \Delta(J_n(1)) + \Delta(J_m(1)), & \text{if } J_n(1) \text{ or } J_m(1) \text{ has a unique Jaconian vertex} \\ \Delta(J_n(1)) + \Delta(J_m(1)) + 1, & \text{otherwise.} \end{cases}$$

Theorem 2.11. If the Jaco Graph $J_n(1)$ has a unique Jaconian vertex (prime Jaconian vertex only) at v_i , then:

- (a) Edge (v_i, v_n) exists and,
- (b) $\Delta(J_n(1)) + d(v_n) = n$.

Proof. The proof follows through contra absurdum. Assume the Jaco Graph $J_n(1)$ has a unique Jaconian vertex v_i . If the edge (v_i, v_n) is undefined then at most, the edge (v_i, v_{n-1})

is defined. When considering vertex v_{i+1} and proceeding with construction per definition, at least the vertex v_{i+1} , can at most, be linked to v_n to have the edge (v_{i+1}, v_n) defined. So, $d(v_{i+1}) \geq d(v_i) = \Delta(J_n(1))$, renders a contradiction on the uniqueness of the Jaconian vertex v_i . Through contra absurdum we conclude that (v_i, v_n) is defined. Hence, result (a) follows.

The Hope subgraph on the vertices $v_{i+1}, v_{i+2}, \dots, v_n$ allows for $d(v_n) = (n - i) - 1$. But with the edge (v_i, v_n) added we have $d(v_n) = (n - i) - 1 + 1 = n - i$.
 $\therefore \Delta(J_n(1)) + d(v_n) = i + (n - i) = n$. Hence, result (b) follows. \square

Note that $\Delta(J_n(1)) + d(v_n) = n \nRightarrow$ uniqueness of the Jaconian vertex.

Theorem 2.12. *Consider the Jaco Graph $J_n(1)$. For $m < i < k \leq n$, the edge (v_m, v_i) can only exist if the edge (v_m, v_{i-1}) exists. Furthermore, if the edge (v_i, v_k) exists then the edges $(v_{i+1}, v_k), \dots, (v_{k-1}, v_k)$ exist.*

Proof. (Part 1): After applying definition 1.1 exhaustively to the vertex v_{m-1} the vertex v_m has attained $d^-(v_m)$. Applying definition 1.1 exhaustively to vertex v_m proceeds by linking the edges $(v_m, v_{m+1}), (v_i, v_{m+2}), \dots$ in such a way as to attain $d(v_m) = \max(\text{abs}(\min(d(v_m)))) \leq m$. So after linking the edge (v_m, v_{i-1}) and, if and only if $d^*(v_m) @ v_{i-1} < m$, can the edge (v_m, v_i) be linked.

(Part 2): If the edge (v_i, v_k) exists then $d^*(v_i) @ v_k \leq i$. So because $i + 1 > i$ it follows that $d^*(v_{i+1}) @ v_k < i + 1$. Hence, by definition 1.1 the edge (v_{i+1}, v_k) exists. \square

Lemma 2.13. *The vertex v_i is the prime Jaconian vertex of a Jaco Graph $J_n(1)$, if and only $d(v_l) \leq d(v_i) = i$ for $l = i + 1, i + 2, \dots, n$.*

Proof. Let the vertex v_i be the prime Jaconian vertex of the Jaco Graph $J_n(1)$, $n \in \mathbb{N}$. If for any $v_l, l = i + 1, i + 2, \dots, n$. we have $d(v_l) > d(v_i)$ then v_i cannot be the prime Jaconian vertex of $J_n(1)$ as it then, contradicts definition 1.3.

Conversely: If $d(v_l) \leq d(v_i) = i$ for $l = i + 1, i + 2, \dots, n$, then it follows from definition 1.2 and 1.3 as well as from property 1, ($d(v_i) > d(v_m) = m$ for all $m \in \{1, 2, 3, \dots, i - 1\}$), that vertex v_i is the prime Jaconian vertex. \square

Theorem 2.14. *If for the Jaco Graph $J_n(1)$, we have $\Delta(J_n(1)) = k$, then the out-degrees of the vertices $v_{k+1}, v_{k+2}, v_{k+3}, \dots, v_n$ are respectively, $\lceil d^+(v_{k+1}) \rceil = (n - k - 1), \lceil d^+(v_{k+2}) \rceil = (n - k - 2), \dots, \lceil d^+(v_{n-1}) \rceil = 1, \lceil d^+(v_n) \rceil = 0$.*

Proof. From definition 1.4 we have that with v_k the prime Jaconian vertex, the Hope subgraph $\mathbb{H}(J_n(1))$, on vertices $v_{k+1}, v_{k+2}, v_{j+3}, \dots, v_n$ is a complete graph. On applying definition 1.1 exhaustively to vertex v_{k+1} it is clearly possible to link $n - (k + 1)$ edges hence, $[d^+(v_{k+1})] = n - (k + 1) = n - k - 1$.

Furthermore, it follows from definition 1.4 that the subgraph on vertices $v_{k+2}, v_{k+3}, v_{j+4}, \dots, v_n$ is a complete graph as well. On applying definition 1.1 exhaustively to vertex v_{k+2} it is clearly possible to link $n - (k + 2)$ edges hence, $[d^+(v_{k+2})] = n - (k + 2) = n - k - 2$.

By repeating the immediate above to vertices v_{k+3}, \dots, v_n and noting that $[d^+(v_n)] = n - n = 0$, the result follows. \square

Theorem 2.15. *If for the Jaco Graph $J_n(1)$, we can express $n = 7 + 3k$, $k \in \{0, 1, 2, \dots\}$, then: $\Delta(J_n(1)) \leq n - (3 + k)$ and, edge $(v_{\Delta(J_n(1))}, v_n)$ exists.*

Proof. For $k = 0$ it follows from the Fisher Algorithm that $\Delta(J_7(1)) \leq 7(3+0)$. Furthermore, since $4 + d^+(v_4) = 7$ the edge (v_4, v_7) exists.

Assume the result holds for $m = 7 + 3k$, $k > 0$, $k \in \{1, 2, \dots\}$ so for the Jaco Graph $J_m(1)$ we have $\Delta(J_m(1)) \leq m - (3 + k)$ and, edge $(v_{\Delta(J)}, v_m)$ exists.

Now consider the Jaco Graph on n vertices with, $n = 7 + 3(k + 1)$. Noting that $\Delta(J_\ell(1)) \leq \ell - 1, \forall \ell \in \mathbb{N}$, it follows that $\Delta(J_n(1)) - \Delta(J_m(1)) < 7 + 3(k + 1) - (7 + 3k) = 3$ hence, $\Delta(J_n(1)) < 3 + \Delta(J_m(1)) < 3 + m - (3 + k) = n - (3 + k)$. So it follows that $\Delta(J_n(1)) \leq n - (3 + k) - 1 = n - (3 + (k + 1))$. \square

3 Fibonacci-Zeckendorf Result

Theorem 3.1. *(Bettina's Theorem): Let $\mathbb{F} = \{f_0, f_1, f_2, f_3, \dots\}$ be the set of Fibonacci numbers and let $n = f_{i_1} + f_{i_2} + \dots + f_{i_r}, n \in \mathbb{N}$ be the Zeckendorf representation of n . Then*

$$d^+(v_n) = f_{i_1-1} + f_{i_2-1} + \dots + f_{i_r-1}.$$

Proof. Through induction we have that first of all, $1 = f_2$ and $d^+(v_1) = 1 = f_1$. Let $2 \leq n = f_{i_1} + f_{i_2} + \dots + f_{i_r}$ and let $k = f_{i_1-1} + f_{i_2-1} + \dots + f_{i_r-1}$. If $i_r \geq 3$, then

$k = f_{i_1-1} + f_{i_2-1} + \dots + f_{i_r-1}$ is the Zeckendorf representation of k , such that induction yields $d^+(v_k) = k = f_{i_1-2} + f_{i_2-2} + \dots + f_{i_r-2}$. Since $k + d^+(v_k) = f_{i_1-1} + f_{i_1-2} + f_{i_2-1} + f_{i_2-2} + \dots + f_{i_r-1} + f_{i_r-2} = f_{i_1} + f_{i_2} + \dots + f_{i_r} = n$, we have $d^+(v_n) = k$.

Finally consider $n = f_{i_1} + f_{i_2} + \dots + f_{i_r}$, $i_r = 2$. Note that $n > 1$ implies that $i_{r-1} \geq 4$ and that the Zeckendorf representation of $n-1$ given by $n-1 = f_{i_1} + f_{i_2} + \dots + f_{i_{r-1}}$. Let $k = d^+(v_{n-1})$. Through induction we have that, $k = f_{i_1-1} + f_{i_2-1} + \dots + f_{i_{r-1}-1}$, and since $i_{r-1} \leq 4$, this is the Zeckendorf representation of k . Accordingly, $d(v_k) = f_{i_1-2} + f_{i_2-2} + \dots + f_{i_{r-1}-2}$, and $k + d^+(v_k) = f_{i_1-1} + f_{i_1-2} + f_{i_2-1} + f_{i_2-2} + \dots + f_{i_{r-1}-1} + f_{i_{r-1}-2} = n-1$. It follows that $d^+(v_n) > k = d^+(v_{n-1})$. Hence, it follows that $d^+(v_n) = k+1 = (f_{i_1-1} + f_{i_2-1} + \dots + f_{i_{r-1}-1}) + f_1 = f_{i_1-1} + f_{i_2-1} + \dots + f_{i_r-1}$. \square

Proposition 3.2. *For Fibonacci numbers $t = f_{n-1}$, $h = f_n$ and $l = f_m$, $t \geq 3$, $h \geq 3$, $l \geq 3$ we have:*

- (a) $\Delta(J_h(1)) = t = f_{n-1}$,
- (b) $\mathbb{J}(J_h(1)) = \{v_t\}$,
- (c) $\Delta(J_{h+l}(1)) = \Delta(J_h(1)) + \Delta(J_l(1))$,
- (d) $\mathbb{J}(J_{h+l}(1)) = \{v_{\Delta(J_h(1)) + \Delta(J_l(1))}\}$.

Proof. From the Fisher Algorithm it follows that for $s = f_2$, $u = f_3$, $w = f_4$ we have:

$$\begin{aligned} f_2 &= 1, \quad d^+(v_s) = 1, \quad \Delta(J_s(1)) = 0 \text{ and } \mathbb{J}(J_1(1)) = \{v_1\}; \\ f_3 &= 2, \quad d^+(v_u) = 1, \quad \Delta(J_u(1)) = 1 \text{ and } \mathbb{J}(J_2(1)) = \{v_1, v_2\}; \\ f_4 &= 3, \quad d^+(v_w) = 2, \quad \Delta(J_w(1)) = 2 \text{ and } \mathbb{J}(J_3(1)) = \{v_2\}. \end{aligned}$$

So assume for $i = f_{k-1}$ and $r = f_k$ we have $d^+(v_r) = f_{k-1}$, $\Delta(J_r(1)) = f_{k-1}$ and $\mathbb{J}(J_r(1)) = \{v_i\}$. So for $q = f_{k+1}$, we have that:

$$f_{k+1} = f_k + f_{k-1} = f_k + d^+(v_r) \Rightarrow v_r$$

is the prime Jaconian vertex of $J_q(1)$ so $\Delta(J_q(1)) = r = f_k$. So result (a) namely, $\Delta(J_h(1)) = t = f_{n-1}$ holds in general.

Bettinas Theorem yields $d^+(v_{r+1}) = d^+(v_r) + 1$. So we have $(f_k + 1) + d^+(v_{r+1}) = (f_k + 1) + d^+(v_r) + 1 = (f_k + d^+(v_r)) + 2 > f_{k+1} + 1$. From Corollary 2.6 it follows that $\mathbb{J}(J_q(1)) = \{v_r\}$. So result (b) namely, $\mathbb{J}(J_h(1)) = \{v_t\}$ holds in general.

The result $\Delta(J_{h+l}(1)) = \Delta(J_h(1)) + \Delta(J_l(1))$ follows from (b) read together with Theorem 2.10.

The result $\mathbb{J}(J_{h+l}(1)) = \{v_{\Delta(J_h(1))+\Delta(J_l(1))}\}$ follows from (c) read together with Theorem 2.10. \square

[Open problem: Note that for $P_0 = \{n \in \mathbb{N} | 1 \leq n \leq 44\}$ we have, $\Delta(J_n(1)) = 3k$ if $n = 5k$. Then for $P_1 = \{n \in \mathbb{N} | 45 \leq n \leq 99\}$ we have, $\Delta(J_n(1)) = 3k + 1$ if $n = 5k$. Then for $P_2 = \{n \in \mathbb{N} | 100 \leq n \leq ???\}$ we have, $\Delta(J_n(1)) = 3k + 2$ if $n = 5k$ and seemingly so on \dots . Find a partitioning $\mathbb{P} = \cup_{i \in \mathbb{N}} P_i$ to close the result.]

[Open problem: Prove Lemma (conjectured) 2.2.]

[Open problem: Prove Theorem (conjectured) 2.10.]

[Open problem: Determine the number of spanning trees of $J_n(1)$.]

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Acknowledgement will be given to colleagues for preliminary peer review and other contributions on the content of this paper during the preprint arXiv publication term:²

[Remark: The concept of Jaco Graphs followed from a dream during the night of 10/11 January 2013 which was the first dream Kokkie could recall about his daddy after his daddy passed away in the peaceful morning hours of 24 May 2012, shortly before the awakening of Bob Dylan, celebrating Dylan's 71st birthday]